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On a Class of Restricted Phase Problems in the Plane

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Certain geometrical necessary and sufficient conditions are developed for the optimal control of a class of second order systems with restrictions on the phase coordinates. The phase constraint set in the plane is assumed to be convex, and under additional hypotheses, it is shown that an optimal trajectory is the one that encloses the least possible area with the optimal trajectory in the unrestricted case.

I. PRELIMINARIES

Some results on the application of Green's theorem to synthesizing optimal controls have been presented in [1], [2], and [3]. In all these, it is assumed that there are no singularities of the integrand of the modified cost functional along the arcs of comparison. When singularities are present, Green's theorem cannot be applied for direct comparison of trajectories. In this paper, we handle the case in which there are at most two singularities. Our main interest is to develop some geometrical necessary and sufficient conditions for finding an optimal trajectory in the presence of restrictions on the phase coordinates. These conditions are utilized to solve some examples in Section V.

To be specific, consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, u) \end{aligned} \qquad \left(\cdot = \frac{d}{dt}\right) \tag{1.1}$$

where u with values in $U \subsetneq R^1$ is a control, $f \in C^1(R^2 \times U^*)$ and $U^* \supseteq U$ is open. Let $G \subsetneq R^2$ be closed, and $x^0, x^1 \in G$. An *admissible control* is a piecewise continuous function defined on some finite $[t_0, t_1]$ with values in U. An *admissible* trajectory $x(t) = (x_1(t), x_2(t))$, $t_0 \leqslant t \leqslant t_1$, joining x^0 to x^1 is the response of (1.1) to an admissible control such that the trajectory lies entirely in G.

Assume that on any given compact set in R^2, the boundary $\partial G(= \mathrm{cl}(G)\backslash\mathrm{int}(G))$ is piecewise C^1, i.e., on that portion of ∂G belonging to this compact set, we have a continuously varying gradient vector everywhere, except at a finite number of

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points. Then the problem is to find an admissible trajectory that minimizes

$$\int_{t_0}^{t_1} f_0(x_1, x_2)\, dt, \tag{1.2}$$

where $f_0 \in C^1(R^2)$ is *nonnegative*, and t_1 is free.

We make the following assumptions, which will be referred to later by their number.

ASSUMPTION 1. $(\partial/\partial x_2)(f_0/x_2) < 0$ almost everywhere in R^2. This is certainly possible if range $(f_0) \subseteq [0, \infty)$ with the zero set of f_0 having measure zero in R^2, and $x_2(\partial f_0/\partial x_2) \leqslant 0$.

ASSUMPTION 2. To avoid several pathological cases, we assume G to be convex.

Let $H^+ = \{(x_1, x_2) \mid x_2 > 0\}$ and $H^- = \{(x_1, x_2) \mid x_2 < 0\}$. We denote the closed upper and lower half planes by $\bar{H}^+$ and $\bar{H}^-$ respectively. Since $\dot{x}_1 = x_2$, along a segment of the trajectory in H^+ (resp. H^-), the x_1-component is increasing (resp. decreasing).

Let C_1 be a trajectory of (1.1) joining x^0 to x^1. By a region we mean an open connected set in the plane. We will only be concerned with bounded regions in this paper. An oriented simple curve Γ joining x^0 to x^1 is said to be *positively* (resp. negatively) *oriented* with respect to C_1 at a region S enclosed by Γ and C_1 if part of Γ with its orientation is one of the components of ∂S when ∂S is traversed in the anti-clockwise (resp. clockwise) direction. If Γ is positively oriented at all regions enclosed by Γ and C_1, we say that Γ is positively oriented with respect to C_1.

DEFINITION 1.1. Suppose C_1 and C_2 are two trajectories joining x^0 to x^1 such that

(a) the regions enclosed by C_1 and C_2 are mutually disjoint; and

(b) C_2 is positively oriented with respect to C_1. Then we say that C_2 is *nicely oriented* with respect to C_1.

Now suppose C_1 and C_2 are two trajectories (joining x^0 to x^1) contained in H^+, and enclosing only one region S in between them. Also assume that C_2 is positively oriented with respect to C_1. Since $\dot{x}_1 > 0$ in H^+, it follows that C_1 is negatively oriented with respect to C_2 at S. Then by Green's theorem,

$$\int_{C_2} f_0\, dt - \int_{C_1} f_0\, dt = \oint_{\partial S} \frac{f_0}{x_2}\, dx_1 = -\int\!\!\int_S \frac{\partial}{\partial x_2}\left(\frac{f_0}{x_2}\right) dx_1\, dx_2, \tag{1.3}$$

which is greater than zero by assumption 1. Thus the trajectories can be compared directly. This can be done if the closure of S does not have any intersection with the x_1-axis, or if the trajectories do not intersect the x_1-axis. In the next few sections, we will consider the cases in which the trajectories have intersection points with the x_1-axis.

We now make a regularity assumption.

ORIENTATION QUALIFICATION 1.2. Let C_1 be an optimal trajectory from x^0 to x^1 in the unrestricted case. We will assume that there is an optimal admissible trajectory C from x^0 to x^1 such that at every region enclosed by C_1 and C, C_1 and C are oppositely oriented with respect to each other. We then say that C obeys the orientation qualification.

That is, if R is a region enclosed by C_1 and C, then there exist A, $B \in \partial R$ such that both C_1 and C are directed from A to B. Although this assumption is difficult to verify, we believe that this is satisfied in most cases. Hereafter, to find an optimal admissible trajectory, we will be limiting ourselves to admissible trajectories that obey the orientation qualification even if we do not explicitly mention this.

Finally we mention that by using the material in Sections II and III, it may be possible to actually show that it is sufficient to consider the admissible trajectories which satisfy the orientation qualification. But in order to do this, there are some hard combinatorial difficulties to be overcome.

II. SOME TOPOLOGICAL PROPERTIES OF THE TRAJECTORIES

From now on, $C_1 = \{x(t)\}$, $t_0 \leqslant t \leqslant t^*$, denotes an optimal trajectory in the unrestricted case from x^0 to x^1 such that $C_1 \not\subseteq G$. Several necessary and sufficient conditions are available to determine C_1 (see [6], [7]). We will consider only the cases in which C_1 has at most two intersection points (i.e., meeting or crossing points) with the x_1-axis. We do not believe that extension to the case where there are more points of intersection with the x_1-axis is a triviality. Let $t_0 \leqslant \theta_1 < \theta_2 \leqslant t^*$ be such that $x(\theta_i)$, $i = 1, 2$, be these intersection points.

ASSUMPTION 3. $x(\theta_i) \in G$ for $i = 1, 2$.

We consider only those trajectories that do not stay at these intersection points for a nonzero duration, without lessening the generality of our method of attack. Also, we consider only the trajectories with no self-intersection, since f_0 is nonnegative.

In the general case, $t_0 < \theta_1 < \theta_2 < t^*$. Since $\dot{x}_1 = x_2$, we can denote the trajectory C_1 in the phase plane by means of the following functions g_1, g_2, and g_3:

$$\begin{aligned} &C_1 \text{ between } x^0 \text{ and } x(\theta_1)\text{: } x_2 = g_1(x_1) \\ &C_1 \text{ between } x(\theta_1) \text{ and } x(\theta_2)\text{: } x_2 = g_2(x_1) \\ &C_1 \text{ between } x(\theta_2) \text{ and } x^1\text{: } x_2 = g_3(x_1). \end{aligned} \tag{2.1}$$

Let a and b be points in the domain of some g_i, $i = 1, 2$, or 3. We call g_i

(i) *Concave* if for each a, b in the domain of g_i, and for each λ, $0 \leqslant \lambda \leqslant 1$,

$$g_i(\lambda a + (1 - \lambda) b) \geqslant \lambda g_i(a) + (1 - \lambda) g_i(b);$$

(ii) *Convex* if $-g_i$ is concave.

We now make an important assumption, which happens to be valid in a number of problems and for a variety of initial and terminal states (e.g., see the examples at the end).

ASSUMPTION 4. Let $i \in \{1, 2, 3\}$. If the graph of g_i is in $\bar{H}^+$, then g_i is concave. If the graph of g_i is in $\bar{H}^-$, then g_i is convex.

Let $x^0 \in \bar{H}^-$. An analogous theory holds if $x^0 \in \bar{H}^+$. Also let $t_0 < \theta_1 < \theta_2 < t^*$, because the cases in which $t_0 = \theta_1$ or $t^* = \theta_2$ can be treated as special cases. After hitting the x_1-axis, the trajectory C_1 can either come back into H^- or go into H^+. This holds true at the second intersection point also. Thus we have the following alternatives ($C_1 = \{x(t) \mid t_0 \leqslant t \leqslant t^*\}$):

$$\begin{array}{rlllll}
\text{I} & x(t) \in H^-, & t_0 \leqslant t \leqslant t^* & & & \\
\text{II} & x(t) \in \bar{H}^-, & t_0 \leqslant t \leqslant t^*, & \text{with} & x(\theta_i) \in x_1\text{-axis}, & i = 1, 2. \\
\text{III} & x(t) \in H^-, & \theta_1 < t < \theta_2; & x(t) \in H^+, & \theta_2 < t \leqslant t^*. & \\
\text{IV} & x(t) \in H^+, & \theta_1 < t < \theta_2; & x(t) \in H^+, & \theta_2 < t \leqslant t^*. & \\
\text{V} & x(t) \in H^+, & \theta_1 < t < \theta_2; & x(t) \in H^-, & \theta_2 < t \leqslant t^*. &
\end{array} \tag{2.2}$$

We will adopt the following convention: if Γ_1 and Γ_2 are two directed simple (nonself-intersecting) paths from A to B in the plane, $\Gamma_1 - \Gamma_2$ is a simple closed curve joining A to A with an obvious induced orientation. $|\Gamma_1 - \Gamma_2|$ denotes the set of points on $\Gamma_1 - \Gamma_2$, but we will not be very rigid about this, and depending upon the context denote this point set also by $\Gamma_1 - \Gamma_2$. If C is a trajectory, C_{AB} denotes that portion of C from A to B. Also $C_{1(AB)}$ denotes that segment of C_1 from A to B.

THEOREM 2.1. *Suppose C_1 obeys assumption 4 and G is convex. Let C be an admissible trajectory that joins x^0 to x^1. If R is a region where C is negatively oriented with respect to C_1, then that part of C_1 which forms the boundary of R is contained in G.*

Proof. Denote by C_{1R} and C_R those portions of C_1 and C forming the boundary of R. The end points A and B of C_{1R} are in G since they belong to C.

If there is no intersection point along C_{1R} with the x_1-axis, the straightline $\overline{AB}$ is in G, since G is convex. Adopting the convention that a simple closed curve is positively oriented if it is described in the counterclockwise direction, $\overline{AB} - C_{1R}$ has positive orientation by assumption 4, and $C_R - C_{1R}$ has negative orientation by hypothesis. By Theorem VII,9,6 of [4] C_{1R} runs inside $|\overline{AB} - C_R|$.[1] Thus the region $\tilde{R}$ enclosed by $\overline{AB}$ and C_R contains C_{1R} (excepting the endpoints). Since G is closed and convex, it contains the closed hull of $\tilde{R}$, and hence $|C_{1R}|$.

Now suppose $x(\theta_i) \in C_{1R}$ for some i. We will make the argument very general by assuming that $x(\theta_i) \in C_{1R}$ for $i = 1, 2$. By assumption 3 and by the convexity of G, the lines $\overline{Ax(\theta_1)}$, $\overline{x(\theta_1)\, x(\theta_2)}$, and $\overline{x(\theta_2)\, B}$ are in G. Arguing as before, $\overline{Ax(\theta_1)} - C_{1(Ax(\theta_1))}$ and $C_R + \overline{Bx(\theta_2)} + \overline{x(\theta_2)\, x(\theta_1)} - C_{1(Ax(\theta_1))}$ have opposite orientations, and hence $C_{1(Ax(\theta_1))}$ is inside $|\overline{Ax(\theta_1)} + \overline{x(\theta_1)\, x(\theta_2)} + \overline{x(\theta_2)\, B} - C_R| \subseteq G$. This implies for $C_{1(Ax(\theta_1))} \subseteq G$. A similar argument applied for $C_{1(x(\theta_1)x(\theta_2))}$ and $C_{1(x(\theta_2)B)}$. ▮

The point of the above theorem is that if an admissible trajectory C is negatively oriented at a region R with respect to C_1, then C_R of C can be replaced by C_{1R} forming an admissible trajectory without increasing the cost. Thus, to find an admissible optimal trajectory, we need to look only among positively oriented (with respect to C_1) admissible trajectories.

LEMMA 2.2. *Any positively oriented trajectory C does not cross C_1.*

Proof. Suppose it does at a point A on C_1. A is connected to a point on the segment of C_1 between A and x^1 by C forming a region S, and to a point on the segment of C_1 between x^0 and A by $-C$ forming a region R. It follows that $C_R - C_{1R}$ and $C_S - C_{1S}$ have opposite orientations, and thus we get the contradiction that C is negatively oriented with respect to C_1 at some region. ▮

THEOREM 2.3. *Consider Case* I *or Case* II *of* (2.2). *Let C be an admissible trajectory that is positively oriented with respect to C_1. Then $C \subseteq \overline{H}^-$.*

Proof. Suppose the trajectory C enters H^+ at some point A. Note that $x^0, x^1 \in \overline{H}^-$ and $\dot{x}_1 = x_2 > 0$ in H^+. In order to reach x^1, C has to hit the x_1-axis again, and let B be the point at which C hits the x_1-axis right after going into H^+. Thus, we have a region R enclosed between the x_1-axis and the segment C_{AB} of C between A and B.

Now we claim that there is a simple path along C and $-C_1$ connecting B to A (see the orientation qualification 1.2). First of all, it is clear that there is a path. If the path is not simple, then it is self-intersecting. But this cannot happen, since

(i) the trajectories are not self-intersecting, and

(ii) by Lemma 2.2, C and C_1 do not cross each other.

[1] By Jordan curve theorem, the plane is divided into two regions by $|\overline{AB} - C_R|$, the inside and the outside, the inside being the bounded component.

Thus we have a bounded connected region $\tilde{R}$ enclosed by C and C_1 and conraining R. For any point $z_0 \in R$, we can see that the argument of $z - z_0$ decrease by 2π radians as z goes around $C_{\tilde{R}} - C_{1\tilde{R}}$, and hence $C_{\tilde{R}}$ is negatively oriented with respect to $C_{1\tilde{R}}$ at $\tilde{R}$. This contradiction establishes the theorem. ▮

Remark 2.4. In cases I and II of (2.2), since $\dot{x}_1 = x_2 \leqslant 0$ in $\bar{H}^-$, the regions enclosed by C_1 and a positively oriented trajectory C are mutually disjoint, noting that $C \subseteq \bar{H}^-$ by Theorem 2.3. Thus, to find an optimal admissible trajectory in these cases, we need only look among nicely oriented admissible trajectories (see definition 1.1).

THEOREM 2.5. *Let $C_1 = \{x(t) \mid t_0 \leqslant t \leqslant t^*\}$ be an optimal trajectory in the unrestricted case, and let $C_1 \subseteq \bar{H}^-$ (case* I *or* II *of* (2.2)). *Also let $x(\theta_i)$, $i \in \{1, 2\}$ be a point of intersection of C_1 with the x_1-axis. Then any positively (or nicely) oriented trajectory coincides with C_1 at $x(\theta_i)$.*

Proof. If $x(\theta_i) = x^0$ or x^1, there is nothing to prove. So assume that $x(\theta_i) \neq x^0$ and $x(\theta_i) \neq x^1$. The x_1-component of x^1 is less than that of x^0, since $\dot{x}_1 = x_2 \leqslant 0$. Let $x(\theta_i) = (a, 0)$ and $l = \{(x_1, x_2) \mid x_1 = a, x_2 < 0\}$. Suppose $x(\theta_i)$ is not a point on C. In order to reach x^1, C crosses l at some point P.

Since the x_1-component along C is decreasing, and since $C \subseteq \bar{H}^-$ (Theorem 2.3), we have $l \cap |C| = \{P\}$. C joins P to x^1, and thus we have a region R enclosed by C and C_1 that contains the open line segment $\overline{x(\theta_i)P}$. It follows that the winding number of any point in the inner domain of $|C_R - C_{1R}|$ with respect to $C_R - C_{1R}$ is -1, since $C_R - C_{1R}$ intersects the directed half ray l negatively, i.e., from left to right (see Chapter VII of [4]). Thus, C is negatively oriented with respect to C_1 at R, contradicting the hypothesis. ▮

Suppose the points P and Q are in $C \cap C_1$. Since $\dot{x}_1 = x_2$ and since C satisfies the orientation qualification 1.2, it follows that if P precedes Q in time on C_1, then P precedes Q in time on C also. Thus $x(\theta_1)$ precedes $x(\theta_2)$ in time on C.

III. Other Cases of (2.2)

Let us now consider cases III, IV, and V of (2.2). We will distinguish between crossing points and meeting points. For example, in case III, $x(t)$ meets the x_1-axis at $x(\theta_1)$, whereas it crosses the x_1-axis at $x(\theta_2)$.

If there are no singular arcs along C_1, then C_1 can be divided into a finite number of segments such that on each segment, $f(x_1, x_2, u)$ (note that $\dot{x}_2 = f(x_1, x_2, u)$) is either maximized or minimized, according to Pontryagin's maximum principle. Since $dx_2/dx_1 = f(x_1, x_2, u)/x_2$, along each segment, either the slope is a maximum, or the slope is a minimum. We now make an important assumption. Note that $C_{1(AB)}$ is the segment of C_1 from A to B.

ASSUMPTION 5. (a) Suppose $x(\theta_1)$ is a crossing point. Then

$$C_{1(x^0x(\theta_1))} \subseteq \bar{H}^- \Rightarrow f \quad \text{is maximized along} \quad C_{1(x^0x(\theta_1))}$$
$$C_{1(x^0x(\theta_1))} \subseteq \bar{H}^+ \Rightarrow f \quad \text{is minimized along} \quad C_{1(x^0x(\theta_1))}$$

(b) Suppose $x(\theta_2)$ is a crossing point. Then

$$C_{1(x(\theta_2)x^1)} \subseteq \bar{H}^- \Rightarrow f \quad \text{is minimized along} \quad C_{1(x(\theta_2)x^1)}$$
$$C_{1(x(\theta_2)x^1)} \subseteq \bar{H}^+ \Rightarrow f \quad \text{is maximized along} \quad C_{1(x(\theta_2)x^1)} .$$

Let us now consider the effect of this assumption in case V of (2.2). At each point along $C_{1(x^0x(\theta_1))}$, we have minimum possible slope, and along $C_{1(x(\theta_2)x^1)}$, we have maximum slope. By Theorem 2.1, to find an optimal admissible trajectory, we need only consider positively oriented admissible trajectories. It follows (as is shown below) that if assumtion 5 is satisfied, any positively oriented admissible trajectory C coincides with $C_{1(x^0x(\theta_1))}$ between x^0 and $x(\theta_1)$, and with $C_{1(x(\theta_2)x^1)}$ between $x(\theta_2)$ and x^1.

To elaborate this point, we will show that C coincides with $C_{1(x^0x(\theta_1))}$ between x^0 and $x(\theta_1)$. Otherwise, C has to depart from $C_{1(x^0x(\theta_1))}$ at some point in H^-. Since $C_{1(x^0x(\theta_1))}$ has minimum possible slope at each point (by assumption 5), C can only go to the left of $C_{1(x^0x(\theta_1))}$. Also, it cannot meet $C_{1(x^0x(\theta_1))}$ again. Thus C intersects the x_1-axis at some point $A \neq x(\theta_1)$ before meeting C_1, because the only way it can meet C_1 before intersecting the x_1-axis is to meet it on $C_{1(x(\theta_2)x^1)}$, but this violates assumption 5 made on $C_{1(x(\theta_2)x^1)}$. Since there is a simple path from A to $x(\theta_1)$ along segments of C and/or $-C_1$, we have a region R at which C is negatively oriented with respect to C_1. This contradiction establishes our point.

Note that the above argument also implies that $C_{1(x^0x(\theta_1))}$ and $C_{1(x(\theta_2)x^1)}$ have to be contained in G. Otherwise, there is no admissible trajectory transferring x^0 to x^1.

Remark 3.1. Similar conclusions can be drawn in cases III and IV also near crossing points. In all cases, by the above argument and by Theorem 2.5, any positively oriented admissible trajectory coincides with C_1 at all $x(\theta_i)$, $i \in \{1, 2\}$. That is, the problem is reduced to that of finding optimal admissible trajectories from x^0 to $x(\theta_1)$, $x(\theta_1)$ to $x(\theta_2)$, and $x(\theta_2)$ to x^1, which are special cases of case II. Thus, by Theorem 2.3 and Remark 2.4, we need to search only among nicely oriented admissible trajectories (with respect to C_1).

IV. PROPERTIES OF OPTIMAL ADMISSIBLE TRAJECTORIES

By the previous section, our problem is reduced to that of finding an optimal admissible trajectory from X to Y where either one or both X and Y are on the x_1-axis and the optimal trajectory in the unrestricted case, namely $C_{1\langle XY\rangle}$ is totally contained either in $\bar{H}^-$ or in $\bar{H}^+$. In fact $C_{1(XY)} \subseteq H^-$ or H^+ except for the endpoints. For notational purposes, let $X = x^0$ and $Y = x^1$. For definiteness, let $C_1 \subseteq \bar{H}^+$, $x^0 \in x_1$-axis and $x^1 \in H^+$. Theorems 4.1 and 4.2 consider only this situation.

Note that the only point of intersection of C_1 with the x_1-axis is x^0. We intend to show that an optimal admissible trajectory C_* is the same as the nicely oriented admissible trajectory that encloses the least possible area with C_1.

We refer to the arguments in the proof of the following uniqueness theorem later.

THEOREM 4.1. *Suppose assumption* 1 *holds. Then C_1 is unique in the following sense: if C_2 is an optimal trajectory in the unrestricted case with only one point of intersection with the x_1-axis, namely x^0, then $C_1 \equiv C_2$.*

Proof. Suppose $C_1 \not\equiv C_2$. Then there exist on the trajectories points A and B such that between A and B, the trajectories are different and form a simple closed curve and a region S inside it. Assume that $C_{2S} - C_{1S}$ has positive orientation.

ARGUMENT M_1. If $A, B \in H^+$, then we have

$$\int_{C_{2(AB)}} f_0\, dt - \int_{C_{1(AB)}} f_0\, dt = \oint_{\partial S} \frac{f_0}{x_2}\, dx_1 = -\iint_S \frac{\partial}{\partial x_2}\left(\frac{f_0}{x_2}\right) dx_1\, dx_2 > 0, \quad (4.1)$$

by assumption 1, contradicting the optimality of C_2. Thus, both A and B cannot belong to H^+.

ARGUMENT M_2. Let $A = x^0$ (see Fig. 1). Let $\epsilon > 0$ be small and t_0 be the

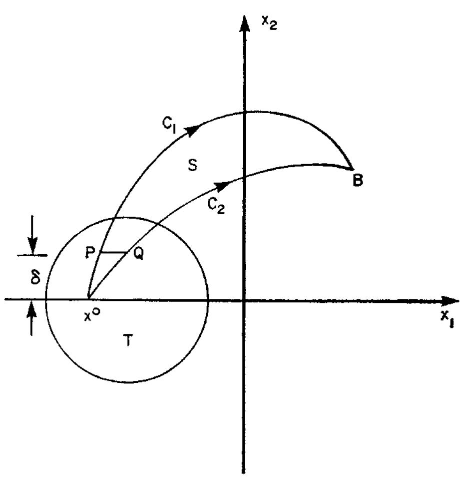

FIGURE 1

initial time. We can choose points P and Q on C_1 and C_2 respectively, such that P and Q have the same x_2-component, and the time taken by the trajectory to reach from x^0 to either point is less than ϵ. Let their x_2-component be δ. Join P to Q by a straight line. Let S be the region enclosed by the curve $PQBP$. By Green's theorem

$$\int_Q^B f_0\, dt - \int_P^B f_0\, dt + \int_P^Q \frac{f_0}{x_2}\, dx_1 = - \iint_S \frac{\partial}{\partial x_2}\left(\frac{f_0}{x_2}\right) dx_1\, dx_2 = L > 0. \tag{4.2}$$

We have,

$$\int_P^Q \frac{f_0}{x_2}\, dx_1 = \int_P^Q \frac{f_0}{\delta}\, dx_1 \leqslant \frac{M}{\delta}(x_1(Q) - x_1(P))$$

where M is the maximum of f_0 on some sufficiently large compact set T containing the curve x^0PQx^0 (see Fig. 1).

But $x_1(t) = x_1(t_0) + \int_{t_0}^t x_2(s)\, ds$. Hence $x_1(Q) < \delta\epsilon + x_1(t_0)$. Also $x_1(P) > x_1(t_0)$. Therefore

$$\int_P^Q \frac{f_0}{x_2}\, dx_1 \leqslant \frac{M}{\delta}\, \delta\epsilon = M\epsilon. \tag{4.3}$$

From (4.2) and (4.3), we get

$$\int_Q^B f_0\, dt - \int_P^B f_0\, dt \geqslant L - M\epsilon. \tag{4.4}$$

The cost to transfer the response from x^0 to either of P and Q is less than $M\epsilon$. So, from (4.4),

$$\int_{C_{2(x^0B)}} f_0\, dt - \int_{C_{1(x^0B)}} f_0\, dt \geqslant L - M\epsilon - M\epsilon.$$

Since ϵ can be made arbitrarily small, it follows that C_2 is not optimal.

If $C_{2S} - C_{1S}$ has negative orientation, we contradict the optimality of C_1, and this establishes the theorem. ∎

THEOREM 4.2. *Let C_* be an admissible trajectory joining x^0 to x^1 such that C_* is nicely oriented with respect to C_1. Suppose assumptions* 1, 2, 3, *and* 4 *hold. Then the following equivalent:*

(i) *C_* is optimal in the restricted case.*

(ii) *C_* encloses the least possible area with C_1 among all nicely oriented admissible trajectories.*

Proof. Assume (i), but suppose C_* does not enclose the least possible area with C_1. Then there is a nicely oriented trajectory C which encloses less area, and C and C_* differ at a region, say S. By Theorem 2.3, both C and C_* are contained in $\bar{H}^+$, and thus $S \subseteq \bar{H}^+$. Further, because C encloses less area with C_1, we can choose S such that C_* is positively oriented at S with respect to C.

If $S \subseteq H^+$, we contradict the optimality of C_* by argument M_1 (see the proof of Theorem 4.1). If $S \subseteq \bar{H}^+$, then there may be a finite number of meetings of either C or C_* or both with the x_1-axis. Argument M_2 shows that these meeting points do not have any effect on the sign of the difference of costs. In other words,

$$\int_{C_{*S}} f_0 \, dt - \int_{C_S} f_0 \, dt > 0.$$

In any case, we contradict the optimality of C_*. Thus (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) by the uniqueness of the nicely oriented admissible trajectory that encloses the least possible area with C_1. ∎

In an analogous manner, we can handle the cases in which $C_1 \subseteq \bar{H}^-$, or $x^1 \in x_1$-axis. By Remark 3.1 and by Theorem 4.2, we get the following recipe for finding an optimal admissible trajectory in all cases:

(i) Check if assumptions 1–5 hold.

(ii) Find a nicely oriented admissible trajectory C_* that encloses the least possible area with C_1.

THEOREM 4.3. *Let C_* be the trajectory obtained by following steps* (i) *and* (ii) *above. Then C_* is optimal in the restricted case. Moreover, any nicely oriented admissible optimal trajectory has to be C_*.*

Proof. We will prove the second statement first. By Theorem 2.1, to find an optimal admissible trajectory, we need only search among positively oriented trajectories. By Theorem 2.5 or assumption 5, depending upon the case, a positively oriented trajectory coincides with C_1 at $x(\theta_i)$, $i = 1$ and/or 2. Thus a positively oriented optimal trajectory C from x^0 to x^1 is also optimal from x^0 to $x(\theta_1)$, $x(\theta_1)$ to $x(\theta_2)$, and $x(\theta_2)$ to x^1. By Theorem 2.3 and Remark 2.4, it follows that C is nicely oriented with respect to C_1, and by Theorem 4.2, C encloses the least possible area with C_1. By the uniqueness of the nicely oriented admissible trajectory that encloses the least possible area with C_1, we have $C = C_*$.

To prove that C_* is optimal in the restricted case, by Theorem 4.2, C_* is optimal from x^0 to $x(\theta_1)$, $x(\theta_1)$ to $x(\theta_2)$, and $x(\theta_2)$ to x^1. This implies that C^* is optimal from x^0 to x^1. ∎

It is obvious that in the place of G being convex, we need only require that each connected component of G be convex. Also, in practice, assumption 4 may be verified by means of a computer plot. We conclude this section by saying that it may be possible to get similar results in some cases where G is not convex, but can be decomposed into a finite union of closed convex sets such that any two in this collection intersect only along their common boundary.

V. Some Examples

Example 1. Consider the time-optimal control of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{5.1}$$

from x^0 to x^1 with $|u| \leqslant 1$, and G as shown in Fig. 2.

C_1 (see [6] or [7]) and an admissible optimal trajectory C_* are shown in Fig. 2. C_* is found by the procedure outlined in Section IV. Between points A and B, we have $x_2 = \frac{1}{8}x_1 + 1$, so that $u = \dot{x}_2 = \frac{1}{8}\dot{x}_1 = \frac{1}{8}x_2$. Similarly, between B and C, $u = -\frac{1}{8}x_2$. In this case, the admissible optimal trajectory can be shown to be unique, by considering the other possibilities in the phase plane.

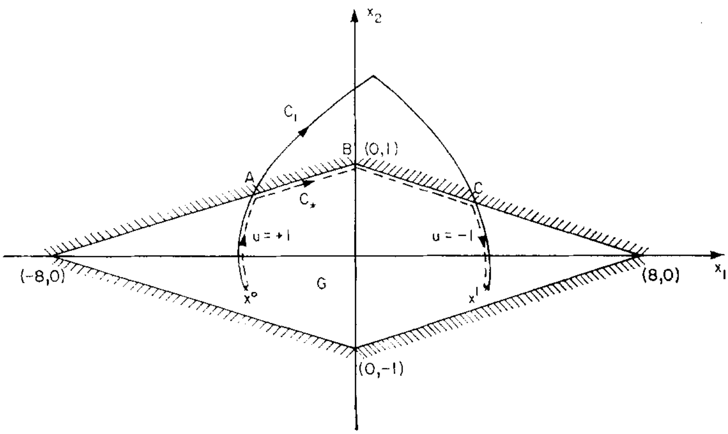

Figure 2

Example 2. Consider the time-optimal control of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \end{aligned} \tag{5.2}$$

from $x(0) = (-1, -0.5)$ to $x(T) = (0, 0)$ with $|u| \leqslant 1$ and $|x_2| \leqslant 0.6$.

C_1, the optimal trajectory in the unrestricted case can be found by applying the maximum principle, and is shown in Fig. 3. The optimal control strategy in the unrestricted case is given by

$$u_1 = \begin{cases} 1 & 0 \leqslant t < 2.1 \\ -1 & 2.1 \leqslant t \leqslant 2.7. \end{cases} \tag{5.3}$$

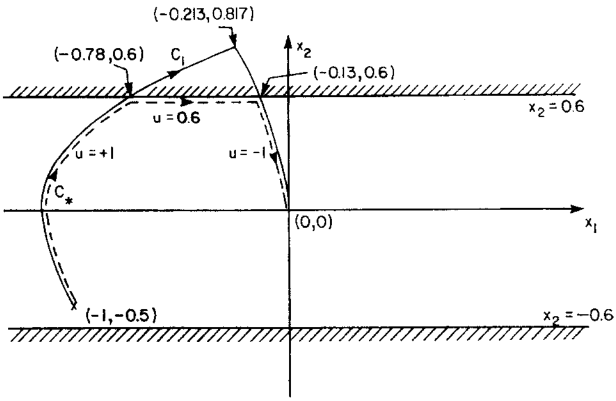

FIGURE 3

To verify assumption 4 on C_1,

$$\frac{dx_2}{dx_1} = \frac{-x_2 \pm 1}{x_2} \Rightarrow \frac{d^2x_2}{dx_1^2} = \frac{\mp dx_2/dx_1}{x_2^2} = \begin{cases} \dfrac{x_2 - 1}{x_2^3} & \text{if} \quad u = 1 \\ \dfrac{-x_2 - 1}{x_2^3} & \text{if} \quad u = -1. \end{cases}$$

Thus, assumption 4 is verified. By our theory, an optimal admissible trajectory C_* can be found, and C_* is shown in Fig. 3. For C_*, the control strategy is given by

$$u_* = \begin{cases} 1 & 0 \leqslant t < 1.32 \\ 0.6 & 1.32 \leqslant t < 2.403 \\ -1 & 2.403 \leqslant t \leqslant 2.873. \end{cases} \tag{5.4}$$

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